TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 334, Number 2, December 1992

ENTROPY FOR CANONICAL SHIFTS

MARIE CHODA

ABSTRACT. For a *-endomorphism σ of an injective finite von Neumann algebra A, we investigate the relations among the entropy $H(\sigma)$ for σ, the relative entropy $H(A|\sigma(A))$ of $\sigma(A)$ for A, the generalized index $\lambda(A, \sigma(A))$, and the index for subfactors. As an application, we have the following relations for the canonical shift Γ for the inclusion $N \subset M$ of type II_1 factors with the finite index $[M : N]$,

$$H(A|\Gamma(A)) \leq 2H(\Gamma) \leq \log \lambda(A, \Gamma(A))^{-1} = 2\log[M : N],$$

where A is the von Neumann algebra generated by the two of the relative commutants of M. In the case of that $N \subset M$ has finite depth, then all of them coincide.

1. INTRODUCTION

The notion of the entropy for *-automorphisms of finite von Neumann algebras is introduced by Connes and Størmer [3]. In the previous paper [2], we defined the entropy for *-endomorphisms of finite von Neumann algebras as an extended version of it. It is possible to define the entropy for a general completely positive linear map α using results in [4] by a similar method. However, the formula of the definition of the entropy for α implies that the entropy is apt to be zero if α^k converges to α when k tends to infinity. A conditional expectation is a typical example of such a map. For that reason, interesting completely positive maps α for us to discuss the entropy are those which have the property that α^k goes away from α as k tends to infinity.

In this paper, we shall study such a class of *-endomorphisms of injective finite von Neumann algebras.

In §3, we introduce, for a *-endomorphism σ of an injective finite von Neumann algebra A, the notion of an n-shift on the tower $(A_j)_j$ of finite dimensional von Neumann subalgebras of A which generates A and we obtain the formula of the entropy $H(\sigma)$ for an n-shift σ.

In the work [9] on the classification for subfactors of the hyperfinite type II_1-factor, Ocneanu introduced a special kind of *-endomorphism which is called the canonical shift on the tower of relative commutants. The *-endomorphism Γ is a generalization of the comultiplication for Hopf algebras and is also considered the canonical shift on string algebras. The *-endomorphism Γ has similar properties to the canonical endomorphism of an inclusion of infinite von Neumann algebras due to Longo [7, 8].

Received by the editors October 30, 1989 and, in revised form, September 7, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L55; Secondary 46L35.

The canonical shift Γ naturally induces a 2-shift for the injective finite von Neumann algebra A generated by the tower $(A_j)_j$ of relative commutants and the entropy $H(\Gamma)$ is determined by the following

$$H(\Gamma) = \lim_{k\to\infty} \frac{H(A_{2k})}{k}.$$

For a *-endomorphism σ of a von Neumann algebra A, the entropy $H(\sigma)$ is a conjugacy invariant, that is, if there is an isomorphism θ of A onto a von Neumann algebra B such that $\theta\sigma = \phi\theta$ for a *-endomorphism ϕ of B, then $H(\sigma) = H(\phi)$. On the other hand, two conjugate *-endomorphisms σ and ϕ of A give two conjugate von Neumann subalgebras $\sigma(A)$ and $\phi(A)$ under automorphisms of A.

In [10], Pimsner and Popa introduced two conjugacy invariants for von Neumann subalgebras. One is the relative entropy $H(A|B)$ for a von Neumann subalgebra B of a finite von Neumann algebra A, which is defined as an extended version of one for finite dimensional algebras due to Connes-Størmer [3]. The other is the generalized index $\lambda(A, B)$, which plays a role like the index for subfactors due to Jones [6]. In fact in the case of factors $B \subset A$, $\lambda(A, B)^{-1}$ is Jones index $[A : B]$. We shall investigate relations among these invariants.

In §4, we restrict our attention to finite dimensional von Neumann algebras. We need these results later. The Jones index for a subfactor N of a finite factor M is given as $1/\tau(e)$ for the projection e of $L^2(M)$ onto $L^2(N)$ where τ is the trace on the basic extension algebra of $N \subset M$. In the case of finite dimensional von Neumann algebras, we shall show that the generalized index $\lambda(\ ,\ )^{-1}$ coincides with Jones index in such a sense.

In §5, we show that in general the following relation holds for an n-shift σ,

$$H(A|\sigma(A)) \leq 2H(\sigma).$$

A condition under which the equality holds is also given.

In §6, we obtain the relation between $H(\sigma)$ and $\lambda(A, \sigma(A))$, the generalized index. We define a locally standard tower for α for an increasing sequence $(A_j)_j$ of finite dimensional von Neumann algebras. The tower $(A_j)_j$ of relative commutants for the inclusion of finite factors $N \subset M$ satisfies this condition. If a *-endomorphism σ of A is an n-shift on a locally standard tower for α which generates A, then we have the following:

$$H(A|\sigma(A)) \leq 2H(\sigma) \leq -\log\alpha \leq \log\lambda(A, \sigma(A))^{-1}.$$

In §7, we shall apply the above results to the canonical shift Γ for the tower of relative commutants. Let $N \subset M$ be type II_1-factors with the finite index. Considering the tower $(M_j)_j$ of factors obtained by iterating Jones basic construction from $N \subset M$, we obtain the increasing sequence $(A_j = M' \cap M_j)_j$ of finite dimensional von Neumann algebras. The *-endomorphisms Γ is defined on the algebra $\bigcup_j A_j$ as a mapping such that $\Gamma(M_k' \cap M_j) = M_{k+2}' \cap M_{j+2}$ for all $k \leq j$. First, we remark that Γ is extended to the trace preserving *-endomorphism of the finite von Neumann algebra $A = \bigcup_j (A_j)''$. The *-endomorphism Γ has an ergodic property that

$$\bigcap_k \Gamma^k(A) = C1,$$

and satisfies all the conditions of the definition for a 2-shift, except one. In order for Γ to satisfy all the conditions for a 2-shift, some additional requirement is needed, and in such a case the generalized index $\lambda(A, \Gamma(A))$ is determined by $[M : N]$,

$$\lambda(A, \Gamma(A))^{-1} = 2[M : N].$$

For example, in the case where $N' \cap M = C1$, Γ is a 2-shift and the following relation holds

$$H(A|\Gamma(A)) \leq 2H(\Gamma) \leq 2\log[M : N].$$

Furthermore, if the inclusion $N \subset M$ has finite depth [9, 13], then we have

$$H(M|N) = H(\Gamma) = \log[M : N].$$

In §8, we discuss conditions for a *-endomorphism σ of a factor M to be extended to an automorphism θ of a factor containing M so that $H(\sigma) = H(\theta)$. If the inclusion $N \subset M$ has finite depth, then Γ is extended to an ergodic *-automorphism Θ which satisfies the following:

$$H(M|N) = H(\Theta) = H(\Gamma) = \log[M : N].$$

2. Preliminaries

In this section, we shall fix the notation and terminology used in this paper. Throughout this section M will be a finite von Neumann algebra with a fixed normal faithful trace τ, $\tau(1) = 1$. We equip M with the structure of a pre-Hilbert space by $\langle x, y\rangle = \tau(xy^*)$. Let $\|x\| = \tau(x^*x)^{1/2}$ and let $L^2(M, \tau)$ by the Hilbert space completion of M. Then M acts on $L^2(M, \tau)$ be the left multiplication. The canonical conjugation on $L^2(M, \tau)$ is denoted by $J = J_M$. It is the conjugate unitary map induced by the involution * on M. For a von Neumann subalgebra N of M, let e_N be the orthogonal projection of $L^2(M, \tau)$ onto $L^2(N, \tau)$. Then the restriction E_N of e_N to M is the faithful normal conditional expectation of M onto N.

The letter η designates the function on $[0, \infty)$ defined by $\eta(t) = -t\log t$. For each k, we let S_k be the set of all families $(x_{i_1, i_2, \dots, i_k})_{i_j \in \mathbf{N}}$ of positive elements of M, zero except for a finite number of indices and satisfying

$$\sum_{i_1, \dots, i_j, \dots, i_k} x_{i_1, \dots, i_k} = 1.$$

For $x \in S_k$, $j \in 1, 2, \dots, k$ and $i_j \in \mathbf{N}$, put

$$x^j_{i_j} = \sum_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_k} x_{i_1, i_2, \dots, i_k}.$$

Let $N_1, N_2, \dots, N_k$ be finite dimensional von Neumann subalgebras of M. Then

$$H(N_1, \dots, N_k) = \operatorname*{Sup}_{x \in S_k} \left[\sum_{i_1, \dots, i_k} \eta\tau(x_{i_1, \dots, i_k}) - \sum_j \sum_{i_j} \tau\eta E_{N_j}(x^j_{i_j}) \right].$$

Let σ be a τ-preserving *-endomorphism of M and N a finite dimensional von Neumann subalgebra of M, then

$$H(N, \sigma) = \lim_{k\to\infty} \tfrac{1}{k} H(N, \sigma(N), \dots, \sigma^{k-1}(N))$$

exists by [2]. The entropy $H(\sigma)$ for σ is defined as the supremum of $H(N, \sigma)$ for all finite dimensional subalgebras N of M.

If there exists an increasing sequence $(N_j)_j$ of finite-dimensional subalgebras which generates M, then by [2]

$$H(\sigma) = \lim_{j\to\infty} H(N_j, \sigma).$$

The relative entropy $H(M|N)$ for a von Neumann subalgebra N of M is defined [10] as an extension form of one [3] by

$$H(M|N) = \operatorname*{Sup}_{x\in S_1} \sum_i [\tau\eta(x_i) - \tau\eta E_N(x_i)].$$

This $H(M|N)$ is a conjugacy invariant for subalgebras of M. Another conjugacy invariant $\lambda(M, N)$ is introduced in [10] as a generalization of Jones index by

$$\lambda(M, N) = \max\{\lambda \geq 0;\, E_N(x) \geq \lambda x,\, x \in M_+\}.$$

For an inclusion $N \subset M$ of finite von Neumann algebras, the von Neumann algebra on $L^2(M, \tau)$ generated by M and $e = e_N$ is called the standard basic extension (or basic construction) for $N \subset M$ and denoted by $M_1 = \langle M, e\rangle$. Then by the properties of $J = J_M$ and $e = e_N$, we have $M_1 = \langle M, e\rangle = JN'J$ [6]. If M_1 is finite and if there is a trace τ_1 on M_1 such that $\tau_1(xe) = \lambda\tau(x)$ for all $x \in M$, then the trace τ_1 is called the λ-Markov trace for $N \subset M$. If $M \supset N$ are factors and there is the λ-Markov trace of M_1 for $N \subset M$, then Jones index $[M : N] = \lambda^{-1}$ [6].

We shall call an increasing sequence $(M_j)_{j\in\mathbf{N}}$ of von Neumann algebras a *standard tower* (cf. [5, 9, 13]) if $M_{j-1} \subset M_j \subset M_{j+1}$ is the basic construction obtained from $M_{j-1} \subset M_j$ for each j.

Let L be a finite factor containing M. We shall call L an algebraic basic construction for the factors $N \subset M$ if there is a nonzero projection $e \in M$ satisfying

(i) $exe = E_N(x)e$ for $x \in M$, and

(ii) L is generated by e and M as a von Neumann algebra.

In this case, there is an isomorphism ϕ of M_1 onto L such that $\phi(e_N) = e$ and $\phi(x) = x$ for all $x \in M$ [11].

We shall call such a projection e a *basic projection* for $N \subset M$ and a decreasing sequence $(N_j)_{j\in N}$ of finite factors a *standard tunnel* (cf. [5, 9, 13]) if $N_{j-1} \supset N_j \supset N_{j+1}$ is an algebraic basic construction for $N_j \supset N_{j+1}$ for each j.

3. Entropy of n-shift

In this section, we shall give the definition of n-shifts and a formula of the entropy for n-shifts. Let A be an injective finite von Neumann algebra with a fixed faithful normal trace τ, with $\tau(1) = 1$. Let $(A_j)_{j=1,2,\dots}$ be an increasing sequence of finite dimensional von Neumann algebras such that $A =$ the weak closure of $\bigcup_j A_j = \{A_j : j\}''$. Assume that σ is a τ-preserving *-endomorphism of A. Then σ is an ultra-weakly continuous, one-to-one mapping with $\sigma(1) = 1$.

Definition 1. Let n be a natural number. A τ-preserving *-endomorphism σ of A is called an n-shift on the tower $(A_j)_j$ for A if the following conditions are satisfied:

(1) For all j and m, the von Neumann algebra $\{A_j, \sigma(A_j), \dots, \sigma^m(A_j)\}''$ generated by $\{\sigma^j(A_j); j = 0, \dots, m\}$ is contained in A_{j+nm}.

(2) There exists a sequence $(k_j)_{j\in\mathbf{N}}$ of integers with the properties

$$\lim_{j\to\infty} \frac{nk_j - j}{j} = 0,$$

and

$$x\sigma^m(y) = \sigma^m(y)x, \qquad \tau(z\sigma^{lk_j}(x)) = \tau(z)\tau(x),$$

for all $l \in \mathbf{N}$, $x, y \in A_j$, $m \in k_j\mathbf{N}$ and $z \in \{A_j, \sigma^{k_j}(A_j), \dots, \sigma^{(l-1)k_j}(A_j)\}''$.

(3) Let E_B be the conditional expectation of A onto a von Neumann subalgebra B of A. Then for $j \geq n$, $E_{A_j}E_{\sigma(A_j)} = E_{\sigma(A_{j-n})}$.

(4) For each j, there exists a τ-preserving *-automorphism or antiautomorphism β of A_{nj+n} such that $\sigma(A_{nj}) = \beta(A_{nj})$.

Remark 1. The number n of an n-shift depends on the choice of the sequence $(A_j)_j$. Every given n-shift can be 1-shift on a suitable tower for the same von Neumann algebra.

Example 1. Let S be the *-endomorphism corresponding to the translation by 1 in the infinite tensor product $R = \bigotimes_{i\in\mathbf{N}}(M_i, \mathrm{tr}_i)$ of the algebra M_i of $m \times m$ matrices with the normalized trace tr_i on M_i for each $i \in \mathbf{N}$. For each j, let $A_j = \bigotimes_{i=1}^{j}(M_i, \mathrm{tr}_i)$. Then for all n, S^n is an n-shift on the tower $(A_j)_j$ for R.

In fact, for an $n \in \mathbf{N}$, let $k_j = [\frac{j}{n}] + 1$. Then $(k_j)_j$ satisfies the following properties (2′) which are stronger than (2):

$$\lim_{j\to\infty} \frac{nk_j - j}{j} = 0,$$

and

$$x\sigma^m(y) = \sigma^m(y)x, \qquad \tau(z\sigma^{lk}(x)) = \tau(z)\tau(x),$$

for all $l \in \mathbf{N}$, $x, y \in A_j$, $k_j \leq k$, $m \in k\mathbf{N}$ and

$$z \in \{A_j, \sigma^k(A_j), \dots, \sigma^{k(l-1)}(A_j)\}''.$$

It is obvious that other conditions are satisfied by S^n.

Example 2. Let $(e_j)_j$ be the sequence of projections with the following properties for some natural number k and $\lambda \in (0, \frac{1}{4}] \cup \{\frac{1}{4}\cos^2(\pi/n); n \geq 3\}$,

(a) $e_ie_je_i = \lambda e_i$ if $|i - j| = k$,

(b) $e_ie_j = e_je_i$ if $|i - j| \neq k$,

(c) $(e_j)_j$ generates the hyperfinite type II_1-factor R,

(d) $\tau(we_i) = \lambda\tau(w)$ for the trace τ of R and a reduced word w on $\{1, e_1, \dots, e_{i-1}\}$.

Let A_j be the von Neumann algebra generated by $\{e_1, \dots, e_j\}$. Then, by [6], A_j is finite dimensional. Let σ be the *-endomorphism of R such that $\sigma(e_i) = e_{i+1}$ [1]. Then σ^n is an n-shift on the tower $(A_j)_j$ of R for all n. In fact, for an $n \in N$, let $k_j = [\frac{j+k}{n}] + 1$. Then $(k_j)_j$ satisfies properties (2′) in

Example 1. The conditions (3) and (4) are satisfied by using results in [6 and 1].

In §7, we shall show that the canonical shift due to Ocneanu is a 2-shift on the tower of relative commutant algebras.

Theorem 1. *If a τ-preserving *-endomorphism σ of A satisfies the condition* (1) *and* (2) *in Definition* 1 *for the tower* $(A_j)_j$ *of* A, *then*

$$H(\sigma) = \lim_{k\to\infty} \frac{H(A_{nk})}{k}.$$

Proof. Theorem 1 is a reformulation of Theorem 9 in [2]. We shall repeat a proof of it for the sake of completeness. Since A is approximately finite dimensional, we have by [2]

$$H(\sigma) = \lim_{j\to\infty}\lim_{k\to\infty}\frac{1}{k}H(A_{nj}, \sigma(A_{nj}), \dots, \sigma^{k-1}(A_{nj})).$$

Hence, by [2 and 3],

$$\begin{aligned} H(\sigma) &\le \lim_j \liminf_k \frac{1}{k}H(\{A_{nj}, \dots, \sigma^{k-j}(A_{nj})\}'', \\ &\qquad\qquad \{\sigma^{k-j+1}(A_{nj}), \dots, \sigma^{k-1}(A_{nj})\}'') \\ &\le \lim_j \liminf_k \frac{1}{k}[H(A_{nj+n(k-j)}) + H(A_{2n(j-1)})] \\ &\le \lim_j \liminf_k \frac{nk}{k}\frac{(A_{nk})}{nk} \\ &= \liminf_k \frac{H(A_{nk})}{k}. \end{aligned}$$

On the other hand, by the condition (2) of n-shift,

$$\tfrac{1}{k}H(A_j, \sigma^{k_j}(A_j), \dots, \sigma^{(k-1)k_j}(A_j)) = H(A_j).$$

Hence by [2 and 3], for a fixed j,

$$\begin{aligned} k_jH(\sigma) &= H(\sigma^{k_j}) \\ &= \lim_i \lim_k \frac{1}{k}H(A_i, \sigma^{k_j}(A_i), \dots, \sigma^{k_j(k-1)}(A_i)) \\ &\ge \lim_k \frac{1}{k}H(A_j, \sigma^{k_j}(A_j), \dots, \sigma^{k_j(l-1)}(A_j)) \\ &= H(A_j). \end{aligned}$$

This implies that

$$H(\sigma) \ge \frac{H(A_j)}{k_j} = \frac{n}{j}H(A_j) - \frac{H(A_j)}{k_j}\frac{nk_j - j}{j}.$$

By the property of k_j, we have

$$H(\sigma) \ge \limsup_j \frac{n}{j}H(A_j) \ge \limsup_j \frac{H(A_{nj})}{j}.$$

Therefore

$$H(\sigma) = \lim_{k\to\infty}\frac{H(A_{nk})}{k}. \quad \square$$

4. Finite dimensional algebras

In this section, M will be a finite dimensional von Neumann algebra and τ a fixed faithful normal trace of M with $\tau(1) = 1$. Then M is decomposed into the direct summands:

$$M = \sum_{l \in K} \bigoplus M_l ,$$

where M_l is the algebra of $d(l) \times d(l)$ matrices and $K = K_M$ is a finite set. Then the vector $d_M = d = (d(l))_{l \in K}$ is called the *dimension vector* of M. The column vector $t_M = t = (t(l))_{l \in K}$ has $t(l)$ as the value of the trace for the minimal projections in M_l, and is called the *trace vector* of τ. Let N be a von Neumann subalgebra of M with $N = \sum_{k \in K_N} \bigoplus N_k$. The *inclusion matrix* $[N \hookrightarrow M] = (m(k, l))_{k \in K_N, l \in K_M}$ is given by the number $m(k, l)$ of simple components of a simple M_l module viewed as an N_k module. Then

$$d_N[N \hookrightarrow M] = d_M \quad \text{and} \quad [N \hookrightarrow M]t_M = t_N .$$

Here we shall give a simple formula for $\lambda(M, N)$.

By the definition of the basic construction of $N \subset M$, there is a natural isomorphism between the centers of N and $\langle M, e \rangle$ via $x \to Jx^*J$. Hence there is a natural identification between the sets of simple summands of N and $\langle M, e \rangle$. We put $K = K_N = K_{\langle M, e \rangle}$.

The following theorem assures that in the case of finite dimensional von Neumann algebras, the constant $\lambda(\cdot)$ plays the same role as the index for finite factors.

Theorem 2. (1) *Assume that there is a trace of $\langle M, e \rangle$ which is an extension of τ. Then*

$$\lambda(\langle M, e \rangle, M)^{-1} = \max_{k \in K} \frac{t_N(k)}{t_{\langle M, e \rangle}(k)} .$$

(2) *If the trace τ of $\langle M, e \rangle$ has the $\tau(e)$-Markov property, then*

$$\lambda(\langle M, e \rangle, M)^{-1} = 1/\tau(e) = \|[N \hookrightarrow M]\|^2 .$$

Proof. (1) Let $(a(l, k))_{l \in K_M, k \in K_{\langle M, e \rangle}}$ be the inclusion matrix $[M \hookrightarrow \langle M, e \rangle]$. Since $[M \hookrightarrow \langle M, e \rangle] = [N \hookrightarrow M]^t$ [6], by the formula in [10],

$$\lambda(\langle M, e \rangle, M)^{-1} = \max_{k \in K} \sum_{l \in K_M} \frac{\min\{a(l, k), d_M(l)\} t_M(l)}{t_{\langle M, e \rangle}(k)} .$$

Since

$$d_M^t = (d_N[N \hookrightarrow M])^t = [M \hookrightarrow \langle M, e \rangle]\, d_N^t ,$$

we have $d_M(l) = \sum_k a(l, k)\, d_N(k)$. It follows that $d_M(l) \geq a(l, k)$ for all l and k. Hence

$$\sum_l \min\{a(l, k), d_M(l)\} t_M(l)$$
$$= \sum_l a(l, k) t_M(l) = ([N \hookrightarrow M] t_M)(k) = t_N(k) .$$

Hence we have

$$\lambda(\langle M, e \rangle, M)^{-1} = \max_{k \in K} \frac{t_N(k)}{t_{\langle M, e \rangle}(k)} .$$

(2) Let $\lambda = \tau(e)$. Then by [6], the following equivalent statements hold:

$$\lambda[N \hookrightarrow M][M \hookrightarrow \langle M, e\rangle]t_N = t_N,$$

and

$$\lambda[M \hookrightarrow \langle M, e\rangle][N \hookrightarrow M]t_M = t_M.$$

Hence we have

$$t_N = [N \hookrightarrow M]t_M = [N \hookrightarrow M][M \hookrightarrow \langle M, e\rangle]t_{\langle M, e\rangle} = \tfrac{1}{\lambda}t_{\langle M, e\rangle}.$$

Since $1/\lambda$ is the Perron-Frobenius proper value of $[N \hookrightarrow M][N \hookrightarrow M]^t$, we have

$$\lambda(\langle M, e\rangle, M)^{-1} = \max_{k\in K} \frac{t_N(k)}{t_{\langle M, e\rangle}(k)} = \frac{1}{\lambda} = \frac{1}{\tau(e)} = \|[N \hookrightarrow M]\|^2. \quad \square$$

Definition 2. Let $N \subset M \subset L$ be an inclusion of finite dimensional von Neumann algebras. Then L is said to be an *algebraic basic construction* for $N \subset M$ if there is a projection e in L satisfying

(a) L is generated by M and e,

(b) $xe = ex$ for an $x \in N$,

(c) If $x \in N$ satisfies $xe = 0$, then $x = 0$,

(d) $exe = E_N(x)e$ for all $x \in M$, ((d) implies (b)).

In this case, there is a *-isomorphism of the basic construction $M_1 = JN'J$ onto L.

We shall call $N \subset M \subset L$ a *locally algebraic extension* of $N \subset M$ if there is a projection $p \in L \cap L'$ which satisfies that the inclusion $M \subset Lp$ is an algebraic basic construction $N \subset M$.

If $L \supset M \supset N$ is a locally standard extension of the inclusion $M \supset N$, we can identity the set K_N with a subset of K_L via the equality $Ne = e(Lp)e$. Under this identification, we have the following:

Proposition 3. *Let $L \supset M \supset N$ be a locally standard extension of $M \supset N$. Then*

$$\lambda(L, M)^{-1} \geq \max_{k\in K_N} \min_{l\in K_L} \frac{t_N(k)}{t_L(l)}.$$

Proof. Let $(a(k, l))_{k\in K_M, l\in K_L} = [M \hookrightarrow L]$. Then by [10],

$$\lambda(L, M)^{-1} \geq \frac{1}{\max_l t_L(l)} \max_l \sum_k \min\{a(k, l), d_M(k)\}t_M(k).$$

Since there is a projection $p \in L \cap L'$ which satisfies that Lp is isomorphic to the basic extension for $N \subset M$, then $[N \hookrightarrow M]^t = [M \hookrightarrow L_p]$. Hence we have, by the same method as in the proof of Theorem 2,

$$\sum_k \min\{a(k, l), d_M(k)\}t_M(k) = t_N(l),$$

for $l \in K_N$, where we consider K_N as a subset of K_L. Thus

$$\lambda(L, M)^{-1} \geq \frac{\max_{l\in K_N} t_N(l)}{\max_{l\in K_L} t_L(l)}. \quad \square$$

Let

$$I(M) = \sum_{l\in K} d(l)t(l) \log \frac{d(l)}{t(l)},$$

where $K = K_M$, $d = d_M$, and $t = t_M$.

Proposition 4. (i) $H(M|N) \leq I(M) - I(N)$,
(ii) $H(\langle M, e\rangle|M) = I(\langle M, e\rangle) - I(M)$,
(iii) $I(M) \leq 2H(M)$ *and the equality holds if and only if* M *is a factor.*

Proof. The inequality (i) is an immediate consequence of the following formula [10]

$$H(M|N) = I(M) - I(N) + \sum_{k,l} d_N(k)m(k,l)t_M(l)\log\min\left\{\frac{d_N(k)}{m(k,l)}, 1\right\},$$

where $(m(k,l))_{k,l} = [N \hookrightarrow M]$.

(ii) By the proof of Theorem 2, $d_M(l) \geq a(l,k)$ for all $l \in K_M$ and $k \in K_{\langle M,e\rangle}$. It follows that $H(\langle M, e\rangle|M) = I(\langle M, e\rangle) - I(M)$.

(iii) Since $d(l)t(l) \leq 1$ for all $l \in K$, we have $I(M) \leq 2H(M)$. The equality holds if and only if $t(l)\,d(l) = 1$, for some l which means that M is factor. □

5. $H(\sigma)$ AND $H(A|\sigma(A))$

In this section we investigate a relation between $H(\sigma)$ and $H(A|\sigma(A))$ for an n-shift σ on the tower $(A_j)_j$ for a finite von Neumann algebra A.

Let $(A_j)_j$ be an increasing sequence of finite dimensional von Neumann algebras. Let $A_j = \sum_{k\in K_j} \bigoplus A_j(k)$ be such a decomposition as in §4, and d_j the dimension vector of A_j. Then we shall say $(A_j)_j$ satisfies *the bounded growth conditions* [2] if the following two conditions are satisfied:

(i) $\sup_j |(K_j)|/j < +\infty$.

(ii) For some m, $A_{j+1}(l)$ contains at most $d_j(k)$ $A_j(k)$-components for all $j \geq m$ where $|(K_j)|$ is the cardinal number of K_j.

For examples, let us consider the two towers which are treated in Examples 1 and 2. Both of them satisfy the bounded growth conditions [2]. We shall discuss another example in §7.

Theorem 5. *Let* σ *be a* τ*-preserving* **-endomorphism of an injective finite von Neumann algebra* A *with a faithful normal trace* τ, $\tau(1) = 1$. *If* σ *is an* n*-shift on the tower* $(A_j)_j$ *for* A, *then* $H(A|\sigma(A)) \leq 2H(\sigma)$.

Furthermore, if the bounded growth conditions are satisfied, for the tower $(A_{nj})_j$,

$$H(A|\sigma(A)) = 2H(\sigma).$$

In order to prove Theorem 5, we need the following:

Lemma 6. *Let* σ *be the same as in Theorem* 5. *If* σ *satisfies the conditions* (1), (3), *and* (4) *in Definition* 1 *for* n, *then*

$$H(A|\sigma(A)) = \lim_{j\to\infty} H(A_{nj+n}|A_{nj}).$$

Proof. By assumptions, the algebra A_{nj+n} contains $\sigma(A_{nj})$. Since two conditional expectations of A_{nj+n} onto A_{nj} and $\sigma(A_{nj})$ are conjugate by the automorphism or antiautomorphism β of A_{nj+n} in the condition (4),

$$H(A_{nj+n}|\sigma(A_{nj})) = H(A_{nj+n}|A_{nj})$$

for all j. On the other hand, A (resp. $\sigma(A)$) is generated by the sequence $(A_{nj+n})_j$ (resp. $(\sigma(A_{nj}))_j$) with the commuting square condition

$$E_{A_{nj}}E_{\sigma(A_{nj})} = E_{\sigma(A_{nj-n})} \quad \text{for all } j.$$

Hence by [10],

$$H(A|\sigma(A)) = \lim_{j\to\infty} H(A_{nj+n}|\sigma(A_{nj})) = \lim_{j\to\infty} H(A_{nj+n}|A_{nj}). \quad \square$$

Proof of Theorem 5. (1) By Lemma 6, Proposition 4 and Theorem 1,

$$\begin{aligned} H(A|\sigma(A)) &= \lim_{j\to\infty} H(A_{nj+n}|A_{nj}) \\ &= \lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k+1} H(A_{nj+n}|A_{nj}) \\ &\le \liminf_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k+1}\{I(A_{nj+n}) - I(A_{nj})\} \\ &= \liminf_{k\to\infty} \frac{1}{k} I(A_{nk+n}) \\ &\le \lim_{k} \frac{1}{k} 2H(A_{nk+n}) \\ &= 2H(\sigma). \end{aligned}$$

(2) In [2], we proved that, if $(A_j)_j$ satisfies the bounded growth conditions, then for the number m in the condition (ii)

$$I(A_j) - I(A_m) = \sum_{i=m+1}^{j} H(A_i|A_{i-1}),$$

and

$$\lim_{j\to\infty} \frac{1}{j}\sum_{k\in K_j} t_j(k)\, d_j(k)\log t_j(k)\, d_j(k) = 0,$$

where t_j is the trace vector of the restriction of τ to A_j.

This implies that

$$\begin{aligned} \lim_{j\to\infty} \frac{I(A_j)}{j} &= \lim_{j\to\infty} \frac{1}{j}\sum_{k\in K_j} t_j(k)\, d_j(k)[\log d_j(k) - \log t_j(k)] \\ &= \lim_{j\to\infty} \frac{2H(A_j)}{j}. \end{aligned}$$

Hence,

$$\begin{aligned} H(A|\sigma(A)) &= \lim_{j} \frac{1}{j}\sum_{i} H(A_{ni+n}|A_{ni}) = \lim_{j} \frac{1}{j} I(A_{nj+n}) \\ &= \lim_{j} \frac{2}{j} H(A_{nj+n}) = 2H(\sigma). \quad \square \end{aligned}$$

By considering the standard tower

$$N \subset M \subset M_1 \subset M_2 \subset \cdots \subset M_n = \langle M_{n-1}, e_{n-1}\rangle \subset \cdots$$

obtained from the pair $N \subset M$ of II_1-factors with $[M:N] < \infty$ by iterating the basic construction, it is proved in [11] that $H(M_n|N) = \log[M_n : N]$ if $H(M|N) = \log[M:N]$. Since the index has the multiplicative property [6], this implies that $H(M_n|N) = nH(M|N)$ if $H(M|N) = \log[M:N]$. The next corollary shows that a similar result holds for the pair $\sigma(M) \subset M$.

Corollary 7. *Let a *-endomorphism σ satisfy the same condition as in Theorem 5. Then for all n,*

$$H(A|\sigma^n(A)) = nH(A|\sigma(A)).$$

Proof. This is an immediate consequence of Theorem 5 and the fact $H(\sigma^n) = nH(\sigma)$ by [2]. □

6. $H(\sigma)$ AND $\lambda(A, \sigma(A))$ FOR n-SHIFT σ

In this section, we shall investigate relations between the entropy $H(\sigma)$ and the constant $\lambda(A, \sigma(A))$ for an n-shift σ of the tower $(A_j)_{j\in N}$ for a finite von Neumann algebra A with a fixed faithful normal trace τ, $\tau(1) = 1$.

Definition 3. We shall call an increasing sequence $(A_j)_j$ of finite dimensional von Neumann subalgebras of a finite von Neumann algebra A with a faithful normal trace τ a *locally standard tower* for α if thee exists a natural number k which satisfies the following conditions:

(1) For a certain central projection $p_{k(j+1)}$ of $A_{k(j+1)}$, the inclusion matrix $[A_{jk} \hookrightarrow A_{k(j+1)}p_{k(j+1)}]$ is the transpose of $[A_{k(j-1)} \hookrightarrow A_{kj}]$, for each j.

(2) If $(t_{k(j-1)}(i))_i$ is the trace vector for the restriction of τ to $A_{k(j-1)}$, then the value of τ of the minimal projections for $A_{k(j+1)}p_{k(j+1)}$ are given by $(\alpha t_{k(j-1)}(i))_i$ for each j.

(3) There is $c > 0$ such that $H(A_{2kj}) \le c - j\log\alpha$ for each j.

We call the number $2k$ a *period* of the locally standard tower.

As examples of locally standard towers, we have the following:

(i) The tower $(A_j)_j$ in Example 1 is obviously a locally standard tower for $1/m$, because the inclusion matrices in each step are all same.

(ii) The standard tower is a locally standard tower for $\|T^tT\|^{-1}$, because the inclusion matrix in the jth step is the transpose of one in the $(j-1)$th step for all j [6]. Hence the tower $(A_j)_j$ is also locally standard if A_{j+1} is a locally algebraic basic extension of $A_{j-1} \subset A_j$.

(iii) The tower $(A_j)_j$ in Example 2 is a locally standard tower for λ, because the central support of e_j in A_j satisfies the conditions (1) and (2) in Definition 3 and the condition (3) is proved by results in §4.2 and §5.1 in [6].

We shall treat another locally standard tower in the next section.

Theorem 8. *Let A be a finite von Neumann algebra with a fixed faithful normal trace τ, $\tau(1) = 1$. Let σ be an n-shift on the locally standard tower $(A_j)_j$ for α with a period $2n$, then*

$$H(A|\sigma(A)) \le 2H(\sigma) \le -\log\alpha \le \log\lambda(A, \sigma(A))^{-1}.$$

Proof. Let d_j and t_j be the dimension vector of A_j and the trace vector of the restriction of τ to A_j, respectively. Let K_j be the set of simple summands of A_j. By the commuting square condition (3) in Definition 1 and [10],

$$\lambda(A, \sigma(A)) = \lim_{j\to\infty} \lambda(A_{nj+n}, \sigma(A_{nj})).$$

Since the conditional expectations $E_{A_{nj}}$ and $E_{\sigma(A_{nj})}$ are conjugate by an automorphism or antiautomorphism β of A_{nj+n}, which satisfies the condition (4),

$$\lambda(A_{nj+n}, \sigma(A_{nj})) = \lambda(A_{nj+n}, A_{nj}).$$

On the other hand, since $(A_j)_j$ is a locally standard tower with a period $2n$, by the same proof as Proposition 3 we have

$$\lambda(A_{nj+n}, A_{nj})^{-1} \geq \max_{k \in K_{nj-n}} \frac{t_{nj-n}(k)}{t_{nj+n}(k)} = \frac{1}{\alpha}.$$

Hence,

$$\log \lambda(A, \sigma(A))^{-1} = \lim_{j \to \infty} \log \lambda(A_{nj+n}, A_{nj})^{-1} \geq -\log \alpha.$$

On the other hand, by the condition (3) of the locally standard tower $(A_j)_j$ for α, we have that

$$H(A_{2nj}) \leq c + j \log \frac{1}{\alpha}.$$

Hence we have by Theorem 1,

$$\log \lambda(A, \sigma(A))^{-1} \geq -\log \alpha \geq 2 \lim_{j \to \infty} \frac{1}{2j} H(A_{2nj}) = 2H(\sigma).$$

Combining with Theorem 5, we have

$$H(A|\sigma(A)) \leq 2H(\sigma) \leq -\log \alpha \leq \log \lambda(A, \sigma(A))^{-1}. \quad \square$$

The above proof shows that under a good condition, $\alpha = \lambda(A, \sigma(A))$. For example, if $(A_j)_j$ is periodic in the sense of [17], the equality holds. We shall show another example in §7.

The author would like to thank F. Hiai for pointing out a mistake in the proof of Theorem 8 in the preliminary version.

Corollary 9. *Let A be an injective finite factor with the canonical trace τ and σ an n-shift of a locally standard tower for A with a period $2n$, then*

$$H(A|\sigma(A)) \leq 2H(\sigma) \leq \log[A : \sigma(A)].$$

Proof. If A is a factor, then $\sigma(A)$ is a subfactor of A, so that, by [10], $[A : \sigma(A)] = \lambda(A, \sigma(A))^{-1}$. Hence we have the corollary. $\square$

In the case that $\sigma(A)$ is a factor, it was determined in [10] when $H(A|\sigma(A)) = \log[A : \sigma(A)]$. In such a case, we have

$$H(A|\sigma(A)) = 2H(\sigma) = \log[A : \sigma(A)].$$

For example, the shifts S in Example 1 and σ for $\lambda > \frac{1}{4}$ in Example 2 satisfy the equality [2]. However, the shifts σ in Example 2 have the following relation, [2]:

$$H(R|\sigma(R)) = 2H(\sigma) < \log[R : \sigma(R)],$$

if $\lambda \leq \frac{1}{4}$.

7. Canonical shift

In [9], Ocneanu defined a very nice *-endomorphism for the tower of the relative commutant algebras for the inclusion $N \subset M$ of type II_1-factors with the finite index.

First we shall recall the definition and main properties of the canonical shift on the tower of relative commutants [9].

Let M be a finite factor with the canonical trace τ and N a subfactor of M such that $[M : N] < +\infty$. Then the basic extension $M_1 = \langle M, e \rangle$ is a II_1-factor with the $\lambda = [M : N]^{-1}$-Markov trace [6] and there is a family $\{m_i\} \subset M$

which forms an "orthonormal basis" in M with respect to the N valued inner product $E_N(xy^*)$ $(x, y \in M)$, that is, each $x \in M$ is decomposed in the unique form as follows [9, 10]:

$$x = \sum_i E_N(xm_i^*)m_i .$$

Iterating the basic construction from $N \subset M$, we have the standard tower

$$M_{-1} = N \subset M_0 = M \subset M_1 = \langle M_0, e_0 \rangle \subset M_2 \subset \cdots .$$

Here, e_j is the projection of $L^2(M_j, \tau_j)$ onto $L^2(M_{j-1}, \tau_{j-1})$ and τ_j is the λ-Markov trace for M_j. Then from the family $(e_j)_j$ the projection $e(n, k)$ is obtained and

$$M_{n-k} \subset M_n \subset M_{n+k} = \langle M_n, e(n, k) \rangle$$

is an algebraic basic extension [9, 11]. Furthermore it is obtained in [9] that the "orthonormal basis" in M_n with respect to M_{n-k}-valued inner product from the family of the basis in $(M_j)_j$.

Let $A_j = M' \cap M_j$ for all j. The antiautomorphism γ_j of $A_{2j} = M' \cap M_{2j}$ defined by

$$\gamma_j(x) = J_j x^* J_j , \qquad x \in A_{2j} ,$$

is called the *mirroring*, where J_j is the conjugate unitary on $L^2(M_j, \tau_j)$. Then for all $x \in M' \cap M_{2j}$, the following expression of the mirrorings is given:

$$\gamma_j(x) = [M_j : M] \sum_i E(em_i^* x)em_i ,$$

where E is the conditional expectation of M_{2j} onto M, e is the projection of $L^2(M_j)$ onto $L^2(M)$ and $(m_i)_i$ a module basis of M_j over M. The expression implies that the mirrorings satisfy the following relation: $\gamma_{j+1} \cdot \gamma_j = \gamma_j \cdot \gamma_{j-1}$; for all $j \geq 1$ on A_{2j-2}. In the view of this relation, the endomorphism Γ of $\bigcup_n A_n$ can be defined by $\Gamma(x) = \gamma_{j+1}(\gamma_j(x))$, for $x \in A_{2j}$. Ocneanu called the endomorphism Γ the *canonical shift* on the tower of the relative commutants. In the case of inclusions of infinite factors, similar *-endomorphisms are investigated by Longo [8]. The mapping Γ has the following properties; for any $k, n \geq 0$ with $n \geq k$, $\Gamma(M_k' \cap M_n) = M_{k+1}' \cap M_{n+2}$.

Now, we shall consider the finite von Neumann algebra A generated by the tower $(A_j)_j$ and extend Γ to a trace preserving *-endomorphism of A as follows.

Since $N \subset M$ are II_1-factors with $[M : N] < +\infty$, there is a faithful normal trace on $\bigcup_j M_j$ which extends the canonical trace τ on M. We denote the trace by the same notation τ.

Although M_{j+1} is defined as a von Neumann algebra on $L^2(M_j, \tau_j)$, each M_j can be considered as von Neumann algebras on the Hilbert space $L^2(M, \tau)$. Hence $\bigcup A_j$ and $\bigcup M_j$ can be considered as von Neumann algebras acting on $L^2(M, \tau)$. Let

$$M_\infty = \left\{ \bigcup_j M_j \right\}'' , \qquad A = \left\{ \bigcup_j A_j \right\}'' .$$

Then M_∞ is a finite factor with the canonical trace which is the extension of τ. We denote it by the same notation τ. Then A is a von Neumann subalgebra of M_∞. Since Γ is a ultra-weakly continuous endomorphism of $\bigcup_j A_j$, Γ is extended to a *-endomorphism of A.

Although, in the case discussed by Ocneanu, for all k, the mirroring γ_k is a trace preserving map thanks to the assumption $N' \cap M = \mathbf{C}1$, in general, the mirrorings are not always trace preserving. However, the canonical shift is always trace preserving.

Lemma 10. *For every k, $\gamma_{k+1} \cdot \gamma_k$ is a τ-preserving isomorphism of $M' \cap M_{2k}$ onto $M_2' \cap M_{2k+2}$. Furthermore, if $E_{A_1}(e_1) = \lambda$ (for example $N' \cap M = \mathbf{C}1$), then γ_j is a trace preserving antiautomorphism of A_{2j} for all j.*

Proof. By the definition, it is obvious that

$$\gamma_{k+1} \cdot \gamma_k(M' \cap M_{2k}) = \gamma_{k+1}(M' \cap M_{2k}) = M_2' \cap M_{2k+2}.$$

In order to prove that $\tau(\gamma_{k+1} \cdot \gamma_k(x)) = \tau(x)$ for all $x \in M' \cap M_{2k}$, it is sufficient to prove that $\tau(\gamma_{k+1}(x)) = \tau(\gamma_k(x))$, for all $x \in M' \cap M_{2k}$. Because of $[M : N] < \infty$, $M' \cap B(L^2(M_k, \tau))$ is a finite factor [6]. Let $(m_i)_i$ be an "orthonormal basis" in M_{k+1} with respect to the M_k-valued inner product $E_{M_k}(xy^*)$, for $x, y \in M_{k+1}$. Every $\xi \in L^2(M_{k+1}, \tau)$ is written in the form $\xi = \sum_i \xi_i m_i$ $(\xi_i \in L^2(M_k, \tau))$. We shall embed an $x \in B(L^2(M_k, \tau))$ into $B(L^2(M_{k+1}, \tau)$ by $x\xi = \sum_i x(\xi_i)m_i$. Then $M' \cap B(L^2(M_k, \tau))$ is considered as a subfactor (with the canonical trace ψ) of the finite factor $M' \cap B(L^2(M_{k+1}, \tau))$ with the canonical trace ϕ. Hence, for an $x \in M' \cap M_{2k} \subset M' \cap B(L^2(M_k, \tau))$, we have

$$\tau(\gamma_k(x)) = \psi(x) = \phi(x) = \tau(\gamma_{k+1}(x)).$$

Assume that $E_{A_1}(e_1) = \lambda = [M : N]^{-1}$. Then by [9 and 10], this implies that

$$\tau_{2j+2}(x) = \tau_{M' \cap B(L^2(M_{j+1}, \tau_{j+1}))}(x)$$

for all $x \in M' \cap M_{2j+2}$, where τ_i (resp. τ_L) is the canonical trace of M_i (resp. factor L). Let $x \in M' \cap M_{2j}$. Since $M' \cap M_{2j} \subset M' \cap M_{2j+2}$,

$$\tau_{2j+2}(\gamma_{j+1}(x)) = \tau_{M' \cap B(L^2(M_{j+1}, \tau_{j+1})}(x).$$

This implies that

$$\tau_{2j+2}(\gamma_{j+1}(x)) = \tau_{2j+2}(x).$$

Thus the mirroring γ_{j+1} is a trace preserving antiautomorphism of A_{2j+2}. □

By Lemma 10, the canonical shift Γ on the tower of the relative commutants $(A_j)_j$ of M is extended to a τ-preserving *-endomorphism of A. We call the *-endomorphism of A the *canonical shift* for the inclusion $M \supset N$ and denote it by the same notation Γ.

We will show the canonical shift Γ is a 2-shift on the tower $(A_j)_j$ for A.

Lemma 11. *Let L be a finite von Neumann algebra with a faithful normal trace τ, $\tau(1) = 1$. If M is a subfactor of L, then*

$$\tau(xy) = \tau(x)\tau(y) \qquad (x \in M, y \in M' \cap L).$$

Proof. Let E be the conditional expectation of L onto M conditioned by τ. For $x \in M$ and $y \in M' \cap L$,

$$E_M(y)x = E_M(yx) = E_M(xy) = xE_M(y),$$

which implies $E_M(y) \in M' \cap M$. Since M is a factor, $E_M(y) = \tau(y)$. Hence

$$\tau(xy) = \tau(E_M(xy)) = \tau(xE_M(y)) = \tau(x)\tau(y). \quad \square$$

Proposition 12. *The canonical shift* Γ *for the inclusion* $N \subset M$ *satisfies the conditions* (1), (2) *and* (3) *for 2-shifts. If* $E_{A_1}(e_1) = [M : N]^{-1}$, *then* Γ *is a 2-shift on the tower* $(A_j)_j$ *for* A.

Proof. Since $[M : N] < +\infty$, for all j, $A_j = M' \cap M_j$ is finite dimensional [6]. For all natural numbers j and k,

$$\Gamma^k(A_j) = \Gamma^k(M' \cap M_j) = M'_{2k} \cap M_{j+2k}.$$

This implies

$$\{A_j, \Gamma(A_j), \dots, \Gamma^m(A_j)\}'' \subset M' \cap M_{j+2m} = A_{j+2m}.$$

For each j, let $k_j = [\frac{j}{2}] + 1$. If $m \geq k_j$, then

$$\Gamma^m(A_j) = M'_{2m} \cap M_{j+2m} \subset A'_j.$$

Combining this with Lemma 11, we have that $(k_j)_j$ satisfies the condition (2) for 2-shifts. It is proved in [13] that $E_{M'_k \cap M_j} E_{M_i} = E_{M'_k \cap M_k}$, for $k \leq i \leq j$. This implies that

$$\begin{aligned} E_{A_j} E_{\Gamma(A_j)} &= E_{M' \cap M_j} E_{M'_2 \cap M_{j+2}} = E_{M' \cap M_j} E_{M_j} E_{M'_2 \cap M_{j+2}} \\ &= E_{M' \cap M_j} E_{M'_2 \cap M_j} = E_{\Gamma(A_{j-2})}. \end{aligned}$$

Hence Γ satisfies (1), (2), and (3) in Definition 1 for $n = 2$.

Assume that $E_{A_1}(e_1) = [M : N]^{-1}$. Then by Lemma 10, the mirroring γ_{j+1} is a trace preserving antiautomorphism of A_{2j+2}. Since $\Gamma(A_{2j}) = \gamma_{j+1}(A_{2j})$, Γ is a 2-shift on the tower $(A_j)_j$. $\square$

Next, we shall show the entropy $H(\Gamma)$ of the *-endomorphism γ of A is always dominated by $\log[M : N]$.

Lemma 13. *Let* $B = A \cap N$ *for von Neumann subalgebras* A *and* N *of a finite von Neumann algebra* M *satisfying the commuting square condition:* $E_A E_N = E_N E_A = E_B$. *Then,* $H(M|N) \geq H(A|B)$, $\lambda(M, N) \leq \lambda(A, B)$.

Proof. By the commuting square condition, we have $E_N(x) = E_B(x)$ for all $x \in A$. Hence

$$\begin{aligned} H(M|N) &= \sup_{x \in S_1 \cap M} \sum_i [\tau\eta E_N(x_i) - \tau\eta(x_i)] \\ &\geq \sup_{x \in S_1 \cap A} \sum_i [\tau\eta E_N(x_i) - \tau\eta(x_i)] = H(A|B), \end{aligned}$$

and

$$\begin{aligned} \lambda(M, N) &= \max\{\lambda\colon E_N(x) \geq \lambda x,\ x \in M_+\} \\ &\leq \max\{\lambda\colon E_B(x) \geq \lambda x,\ x \in A_+\} = \lambda(A, B). \end{aligned}$$

Let B and C be the von Neumann subalgebras of A defined by

$$B = \left(\bigcup_j (M'_1 \cap M_j)\right)'', \qquad C = \left(\bigcup_j (M'_2 \cap M_j)\right)''.$$

Theorem 14. *Let* Γ *be the canonical shift for the inclusion* $N \subset M$ *of type* II_1*-factors with* $[M : N] < \infty$. *Then*

$$H(\Gamma) = \lim_{k\to\infty} \frac{H(M' \cap M_{2k})}{k}.$$

If $E_{A_1}(e_1) = [M : N]^{-1}$, *then*

$$H(A|C) \leq 2H(\Gamma) \leq \log\lambda(A, C)^{-1} = 2H(M|N) = 2\log[M : N].$$

Proof. The shift Γ satisfies conditions (1) and (2) for 2-shifts. Hence by Theorem 1,

$$H(\Gamma) = \lim_{k\to\infty} \frac{H(A_{2k})}{k}.$$

Assume that $E_{A_1}(e_1) = [M : N]^{-1}$. Then the canonical shift Γ is a 2-shift on the tower $(A_j)_j$ of the relative commutants of M by Proposition 12. For the projection e_j of $L^2(M_j, \tau)$ onto $L^2(M_{j-1}, \tau)$, let p_j be the central support of e_j in A_j. Then, for all $j \geq 1$,

$$A_{j-1} \subset A_j \subset A_{j+1}p_{j+1}$$

is an algebraic basic extension for $A_{j-1} \subset A_j$ and the trace vectors of A_{j-1} and A_{j+1} satisfy the condition (2) in Definition 3 for $\lambda = [M : N]^{-1}$, [5, 9, 13, 17]. On the other hand, [10, Theorem 4.4] assures that for all j,

$$2H(M' \cap M_j) \leq H(M_j|M).$$

Since

$$H(M_j|M) \leq \log[M_j : M] = -j\log\lambda,$$

by [10 and 11], the condition (3) in Definition 3 for (A_j) is satisfied. Hence the sequence $(A_j)_j$ is a locally standard tower for λ^2 with period 4. Hence, by Theorem 8,

$$H(A|\Gamma(A)) \leq 2H(\Gamma) \leq 2\log[M : N] \leq \log\lambda(A, \Gamma(A))^{-1}.$$

Since $\Gamma(M_k' \cap M_j) = M_{k+2}' \cap M_{j+2}$, we have $\Gamma(A) = C$. Hence,

$$H(A|C) \leq 2H(\Gamma) \leq 2\log[M : N] \leq \log\lambda(A, C)^{-1}.$$

Every factor M_j can be considered as a von Neumann algebra on $L^2(M, \tau)$ by Jones' method [6]. Then as von Neumann algebras on $L^2(M, \tau)$, for all j,

$$E_{M'\cap M_j}E_{M_2'} = E_{M_2'}E_{M'\cap M_j} = E_{M_2'\cap M_j}$$

Since A is generated by the tower $(M'\cap M_j)_j$ and C is generated by the tower $(M_2' \cap M_j)_j$, it follows that $E_A E_{M_2'} = E_C$, where all algebras are considered as von Neumann subalgebras of a finite factor M' on $L^2(M, \tau)$. By Lemma 13, this implies $\lambda(M_2', M') \leq \lambda(A, C)$. Since M and N are factors, $\lambda(M, N)^{-1} = [M : N]$. On the other hand, Jones proved that $M' \supset M_2'$ are finite factors with $[M' : M_2'] = [M_2 : M] = [M : N]^2$. Hence

$$\lambda(A, \Gamma(A))^{-1} = \lambda(A, C)^{-1} = 2[M : N].$$

The condition that $E_{A_1}(e_1) = [M : N]^{-1}$ is equivalent to $H(M|N) = \log[M : N]$ [10]. Hence we have

$$H(A|C) \leq 2H(\Gamma) \leq \log\lambda(A, C)^{-1} = 2\log[M : N] = 2H(M|N). \quad \square$$

The above simple proof, where the condition (3) in Definition 3 for the sequence $(A_j)_j$ was used, was indicated by F. Hiai.

As an immediate consequence, we have

Corollary 15. *Under the same conditions as in Theorem* 14, *let* A *be a factor. Then*

$$H(A|C) \leq 2H(\Gamma) \leq 2\log[A:B] = 2\log[M:N].$$

Corollary 16. *Let* Γ *be the canonical shift for the inclusion* $N \subset M$ *of type* II_1*-factors with* $[M:N] < \infty$. *If* $N' \cap M = \mathbf{C}1$, *then*

$$H(\Gamma) \leq H(M|N) = \log[M:N].$$

For a pair $N \subset M$ of hyperfinite type II_1-factors with $[M:N] < \infty$, Popa says that $N \subset M$ has the *generating property* if there exists a choice of the standard tunnel of subfactors $(N_j)_j$ such that M is generated by the increasing sequence $(N_j' \cap M)_j$.

Corollary 17. *Assume that* $N \subset M$ *has the generating property. If* $E_{N'\cap M}(e_0) = [M:N]^{-1}$, *then*

$$H(M|N) = H(\Gamma) = \log[M:N].$$

Proof. By [6], we consider all M_j as factors acting on $L^2(M, \tau)$. Let J be the canonical conjugation on $L^2(M, \tau)$. For each j, let $N_j = JM_j'J$. Then the mapping Φ defined by $\Phi(x) = JxJ$ is a trace preserving anti-isomorphism [13] such that $\Phi(A) = (\bigcup_j (N_j' \cap M))''$ and $\Phi(B) = (\bigcup_j (N_j' \cap N))''$ because $E_{N'\cap M}(e_0) = [M:N]^{-1}$. Although, the tunnel of subfactors is not uniquely determined, the pair of algebras of relative commutants is unique up to isomorphism [13], that is, let $M \supset N \supset N_1 \supset \cdots$ and $M \supset N \supset P_1 \supset \cdots$ be two choices of the standard tunnels, then there exists a trace preserving isomorphism Ψ such that

$$\Psi\left(\left(\bigcup_j (N_j' \cap M)\right)''\right) = \left(\bigcup_j (P_j' \cap M)\right)''$$

and

$$\Psi\left(\left(\bigcup_j (N_j' \cap N)\right)''\right) = \left(\bigcup_j (P_j' \cap N)\right)''.$$

Since $N \subset M$ has the generating property, we have a trace preserving antiautomorphism of M onto A which transpose N_1 onto C. Hence $H(A|C) = H(M|N_1)$. If $E_{N'\cap M}(e_0) = [M:N]^{-1}$, then $H(M|N_1) = \log[M:N_1]$ [11]. Hence by Theorem 14,

$$H(M|N) = H(\Gamma) = \log[M:N]. \quad \square$$

As a sufficient condition for the two assumptions in Corollary 17, Ocneanu [9] introduced the following notion for a pair $N \subset M$ with $N' \cap M = \mathbf{C}1$, and Popa [13] extended it to general cases. The inclusion $N \subset M$ of type II_1-factors with $[M:N] < +\infty$ is said to have the *finite depth* if $\sup_j(k_j) < +\infty$, where k_j is the cardinal number of simple summands of $M' \cap M_j$.

Remark 18. If the inclusion $N \subset M$ of type II_1-factors with the finite index and finite depth, then the tower $(A_j)_j$ of relative commutants satisfies the bounded growth conditions.

If an inclusion $N \subset M$ has the finite depth, then $E_{N'\cap M}(e_0) = [M:N]^{-1}$ and $N \subset M$ has the generating property [13]. Hence we have

Corollary 19. *Let $N \subset M$ be type II_1-factors with the finite index and the finite depth. Let Γ be the canonical shift for $N \subset M$. Then*

$$H(M|N) = H(\Gamma) = \log[M : N].$$

Remark 20. In Corollary 18, the shift Γ is considered as an *-endomorphism of the algebra A generated by the tower $(A_j)_j$ of the relative commutants of M. Since $N \subset M$ has the finite depth, the shift Γ induces a trace preserving *-endomorphism of M which sending M to the subfactor P in such a way that $P \subset N \subset M$ is the algebraic basic extension for $P \subset N$. Then the *-endomorphism of M has the same property as Γ.

In the rest of this section, we shall show that the canonical shift has an ergodic property, which is similar to the canonical endomorphism in [7]. Therefore the canonical shift is a shift in the sense due to Powers [14].

Proposition 21. *Let $N \subset M$ be type II_1-factors with the finite index. Then the canonical shift Γ for $N \subset M$ satisfies that*

$$\bigcap_k \Gamma^k(A) = \mathbf{C}1.$$

Proof. The von Neumann algebra A is contained in the type II_1-factor $M_\infty = (\bigcup_j M_j))''$ with the canonical trace τ which is the extension of τ. Let take an $x \in \bigcap_x \Gamma^k(A)$. For any $\varepsilon > 0$, there exists an integer k such that $\|x - x_k\|_2 < \varepsilon$ for some $x_k \in A_k$. Let E be the conditional expectation of M_∞ onto M_k. Since $x \in \Gamma^k(A) \subset M_k' \cap M_\infty$, for any $y \in M_k$, $E(x)y = E(xy) = yE(x)$. This implies $E(x) \in M_k \cap M_k'$, that is, $E(x) = \tau(x)$. On the other hand, $x_k \in M_k$. Hence

$$\|x - \tau(x)\|_2 \leq \|x - x_k\|_2 + \|x_k - E(x)\|_2 < 2\varepsilon.$$

This means, $x \in \mathbf{C}1$. □

8. Extension of canonical shift

In this section, we shall show that the canonical shift Γ is extended to an ergodic *-automorphism Θ of a larger von Neumann algebra in such a way that $H(\Gamma) = H(\Theta)$.

Let $N \subset M$ be type II_1-factors with $[M : N] < \infty$. Let

$$M_{-1} = N \subset M = M_0 \subset M_1 = \langle M, e \rangle \subset \cdots \subset M_j = \langle M_{j-1} e_{j-1} \rangle \subset \cdots$$

be the standard tower obtained from $N \subset M$. Let M_∞ be the finite factor generated by the tower $(M_j)_j$.

Proposition 22. *Let $N \subset M$ be type II_1-factors with the finite index and τ the canonical trace of M. Let σ be a *-isomorphism of M onto N. Then the following statements are equivalent:*

(1) *There exists a *-isomorphism σ_1 of M_1 onto M such that for all $x \in M$, $\sigma_1(x) = \sigma(x)$.*

(2) *There exists a projection $e \in M$ such that $\sigma(N) = \{e\}' \cap N$ and $E_N(e) = \lambda 1 = [M : N]^{-1}$.*

(3) *There exists a projection $e \in M$ such that for all $y \in N$, $eye = E_{\sigma(N)}(y)e$, $\tau(ey) = \lambda\tau(y)$, and M is generated by N and e as a von Neumann algebra.*

(4) *There exists an automorphism* Θ *on* M_∞ *such that for all* $x \in M$ *and all* j, $\Theta(x) = \sigma(x)$ *and* $\Theta(e_j) \in M_j$.

(5) *The decreasing sequence* $M \supset N \supset \sigma(N) \supset \cdots \supset \sigma^j(N) \supset \cdots$ *is a standard tunnel.*

Proof. (1) $\Rightarrow$ (2). Let $e = \sigma_1(e_0)$, where e_0 is the projection of $L^2(M, \tau)$ onto $L^2(N, \tau)$. Since σ must be τ-preserving, for all $x \in M$,

$$\sigma(E_M(x)) = E_{\sigma(M)}(\sigma(x)).$$

By [6], $E_M(e_0) = \lambda 1$ and $N = \{e_0\}' \cap M$. Hence (2) holds.

(2) $\Rightarrow$ (3). The projection e in (2) satisfies that $eye = E_{\sigma(N)}(y)e$ for all $y \in N$ and $M = \{N, e\}''$ by [11]. If $y \in N$, then

$$\tau(ey) = \tau(E_{\sigma(N)}(y)e) = \tau(E_{\sigma(N)}(y)E_N(e)) = \lambda\tau(y).$$

(3) $\Rightarrow$ (1). We put

$$\sigma_1\left(\sum_{I=1}^{k} a_i e_0 b_i\right) = \sum_{i=1}^{k} \sigma(a_i)e\sigma(b_i),$$

for $a_i, b_i \in M$. The map σ is a well-defined *-homomorphism.

In fact, assume $z = \sum_i a_i e_0 b_i = 0$. Since σ is trace preserving,

$$\begin{aligned}
\|z\|_2^2 &= \sum_{i,j} \tau(b_i^* e_0 a_i^* a_j e_0 b_j) \\
&= \sum_{i,j} \tau(e_0 E_N(a_i^* a_j) E_N(b_j b_i^*)) \\
&= \lambda \sum_{i,j} \tau(E_{\sigma(N)}(\sigma(a_i^*)\sigma(a_j)) E_{\sigma(N)}(\sigma(b_j)\sigma(b_i^*))) \\
&= \sum_{i,j} \tau(e E_{\sigma(N)}(\sigma(a_i)^*\sigma(a_j))\sigma(b_j)\sigma(b_i^*)) \\
&= \left\| \sum_i \sigma(a_i)e\sigma(b_i) \right\|_2^2 .
\end{aligned}$$

Thus σ_1 is extended to a *-isomorphism of M_1 onto M. By the definition, for all $a \in M$, $\sigma(a) = \sigma_1(1) = \sigma_1(a) = \sigma_1(1)\sigma(a)$ and $e\sigma_1(1) = \sigma_1(e_0) = \sigma_1(1)e$, because σ_1 is a *-isomorphism of M_1 onto M. Since the factor M is generated by N and e, the projection $\sigma_1(1) = 1$. Hence for all $x \in M$, $\sigma_1(x) = \sigma_1(x1) = \sigma(x)$.

(1) $\Rightarrow$ (4). Let us consider the *-isomorphism σ_1 of M_1 onto M such that $\sigma_1(x) = \sigma(x)$ for all $x \in M$. Then the projection $e_0 \in M_1$ satisfies that $\sigma_1(M) = N = e_0' \cap M$ and $E_M(e_0) = \lambda 1$. Hence the above discussion implies that there exists a *-isomorphism σ_2 of M_2 onto M_1 such that $\sigma_2(x) = \sigma_1(x)$ for $x \in M_1$. Iterating this method, we have the sequence $(\sigma_j)_j$ of *-isomorphisms of M_j onto M_{j-1} such that $\sigma_j(x) = \sigma_{j-1}(x)$ for $x \in M_{j-1}$. For any $y \in \bigcup_j M_j$, let $\Theta(y) = \sigma_j(y)$ if $y \in M_j$. Then θ is extended to the (we denote it by the same notation Θ) mapping of M_∞. The mapping Θ is an automorphism and $\Theta'(x) = \tau(x)$ for $x \in M_\infty$ and $\Theta(e_j) = \sigma(e_j) \in M_j$.

(4) $\Rightarrow$ (1). The automorphism Θ satisfies that $\Theta(M) = N$ and $\Theta(e_0) \in M$. Hence Θ is an automorphism of M_1 onto M such that $\Theta(x) = \sigma(x)$ for $x \in M$.

(3) $\Rightarrow$ (5). Let us take such a projection e as in (3). If $z \in \sigma(N)$ satisfies $ze = 0$, then $0 = \|ez\|_2 = \lambda\|z\|_2$. Hence $z = 0$. Clearly, M is an algebraic basic extension for $\sigma(N) \subset N$. Let $\sigma^i(e) = e_{-i}$ and $N_i = \sigma^i(N)$. Then $N_i \supset N_{i+1} \supset N_{i+2}$ is an algebraic basic extension for $N_{i+1} \supset N_{i+2}$.

(5) $\Rightarrow$ (3). Since the tunnel is standard, there is the basic projection $e \in M$ for $\sigma(N) \subset N$. The projection e satisfies the conditions (3). □

Definition 4. Let σ be a *-isomorphism of a type II_1-factor M onto a subfactor N with the finite index. If σ satisfies the equivalent conditions in Proposition 22, then we call σ a basic *-endomorphism for the inclusion $N \subset M$.

Let σ be a basic *-endomorphism of the inclusion $N \subset M$ of type II_1-factors with the finite index. Let $P_j = M \cap \sigma^j(M)'$. Then $(P_j)_j$ is an increasing sequence of finite dimensional von Neumann algebras. Let P be the von Neumann algebra generated by $(P_j)_j$. Then P is a von Neumann subalgebra of M and we have the following

Proposition 23. *Let σ be a basic *-endomorphism for the inclusion $N \subset M$ of type II_1-factors with the finite index. Then,*

$$H(\sigma) = \lim_{k\to\infty} \frac{H(M \cap \sigma^k(M)')}{k}.$$

Assume that $E_{N'\cap M}(e) = [M : N]^{-1}$ for a basic projection of $\sigma(N) \subset N$. Then σ^m is a m-shift on the tower $(P_j)_j$ for P for all even number m and satisfies the following relations. For all even m,

$$H(P|\sigma^m(P)) \le 2mH(\sigma) \le \log\lambda(P, \sigma^m(P))^{-1} = m\log[M : N].$$

Proof. The condition (1) is obviously satisfied. For every j, put $k_j = [\frac{j}{n}] + 1$. Then by Lemma 11, (2) for n-shift is satisfied. Hence we have the first equality. Since $(\sigma^j(M))_j$ is a standard tunnel, $(\sigma^{jn}(M))_j$ is a standard tunnel. Hence the commuting square condition (3) is satisfied [13]. We take the mirroring γ defined by the conjugation on $L^2(\sigma^{n(j+1)}(M))$. Then by a similar method as in the proof of Lemma 10, γ is the trace preserving antiautomorphism of $P_{2n(j+1)}$ such that $\gamma(P_{2nj}) = \sigma^{2n}(P_{2nj})$, because σ is a basic *-endomorphism. Hence σ^{2n} is an $2n$-shift on the tower $(P_j)_j$ for P, and by Theorem 8 and [2], for all n,

$$H(P|\sigma^{2n}(P)) \le 2H(\sigma^{2n}) = 4nH(\sigma).$$

Let p_j be the central support of the projection e_{-j} in P_j which satisfies that $\sigma^j(M) = e'_{-j} \cap \sigma^{j-2}(M)$. Then the inclusion $P_{j+1} \subset P_{j+2}p_{j+2}$ is an algebraic basic construction corresponding to $P_j \subset P_{j+1}$ via $P_{j+1} \simeq P_{j+2}p_{j+2}$. This means that $(P_j)_j$ is a locally standard tower with a period 2, for $\lambda = [M : N]^{-1}$ that is, with every even number as a period. Hence

$$2H(\sigma^m) \le \log\lambda(P, \sigma^m(P))^{-1},$$

for all even m. Since $E_P E_{\sigma^n(M)} = E_{\sigma^n(P)}$, by Lemma 13,

$$\log\lambda(P, \sigma^n(P))^{-1} \le \log\lambda(M, \sigma^n(M))^{-1} = \log[M : \sigma^n(M))] = n\log[M : N].$$

Thus we have the stated inequality. □

Corollary 24. *Let* σ *be the same as in Proposition* 23. *Then*

$$2H(\sigma) \leq \log[M : N].$$

Furthermore, if the inclusion $N \subset M$ *has finite depth, then*

$$H(M|N) = 2H_M(\sigma) = 2H(\sigma) = \log[M : N],$$

where $H_M(\sigma)$ *is the entropy of* σ *as a* **-endomorphism of* M.

Proof. The first inequality is clear by Proposition 23. Assume that the inclusion $N \subset M$ has finite depth. Then it is proved in [13] that there exists a choice of the standard tunnel $(N_i)_i$ such that M is generated by $\{N_i' \cap M\}_i$. Since $(\sigma^i(M))_i$ is also a standard tunnel of subfactors, there exists a trace preserving *-isomorphism of M onto P carrying N onto $\sigma(P)$ [13]. The finite depth assumption implies that $E_{N' \cap M}(e_N) = 1/[M : N]$ by [13]. Hence $\log[M : N] = H(M|N)$ by [10]. On the other hand, $H(M|\sigma^n(M)) = H(P|\sigma^n(P))$ for all n, because σ is a trace preserving *-endomorphism of M. Hence

$$H(M|\sigma^n(M)) = \log[M : \sigma^n(M)] = n \log[M : N].$$

By Proposition 23,

$$H(M|N) = 2H_M(\sigma) = 2H(\sigma) = \log[M : N]. \quad \square$$

As an example of a basic *-endomorphism, we have the *-endomorphism σ in Example 2.

We shall show that another typical example of a basic *-endomorphism is the canonical shift on the tower of relative commutants in §7.

Proposition 25. *Let* $M \supset N$ *be type* II_1*-factors with the finite index and finite depth. Then the canonical shift* Γ *for the inclusion* $M \supset N$ *is a basic* **-endomorphism of* $A = (\bigcup_j (M' \cap M_j)_j)''$.

Proof. If $M \supset N$ has finite index and finite depth, then A is a finite factor which is anti-isomorphic to M. Let C be the subfactor $\Gamma(A)$. Then $[A : C] = [M : N]^2$. To prove that Γ is the basic *-endomorphism of A, we have to show the existence of a projection in A which satisfies the statement (2) in Proposition 22. Let f be a projection in M_4 such that M_4 is generated by M_2 and f. Then

$$M_4' \cap M_j = \{f\}' \cap M_2' \cap M_j$$

for all $j \geq 4$. By the definition of A and the property of Γ, $\Gamma(C) = \{f\}' \cap C$. Since f is the basic projection for the standard tower $M \subset M_2 \subset M_4$ and $N \subset M$ has finite depth, by [13],

$$E_C(f) = [M : N]^2 = [A : C]. \quad \square$$

In [2], we proved that some kinds of *-endomorphisms are extended to ergodic *-automorphisms of larger algebras with same values as entropies. Here we shall show this also holds for the canonical shifts.

Let R be the von Neumann algebra generated by the standard tower obtained from $A \supset \Gamma(A)$. Since Γ is a basic *-endomorphism of A, there exists a *-automorphism of R which is an extension of Γ. We denote it by Θ.

Theorem 26. *Let $N \subset M$ be type II_1-factor with finite index. Then the automorphism Θ induced by the canonical shift Γ for the inclusion $N \subset M$ is ergodic. If $N \subset M$ has finite depth*

$$H(M|N) = H(\Theta) = H(\Gamma) = \log[M : N].$$

Proof. Let us take an $x \in R$ such that $\Theta(x) = x$. By considering the standard tunnel obtained through Γ,

$$\cdots \subset N_k = M_{-k} \subset \cdots \subset N_1 = N = M_{-1} \subset M_0 = M \subset M_1 \subset \cdots \subset M_j \subset \cdots$$

we observe that R is generated by $\bigcup_{k,j}(N'_k \cap M_j)$. Then for any $\varepsilon > 0$ there are k and j such that $\|x - x'\|_2 < \varepsilon$ for some $x' \in N'_k \cap M_j$. Since Θ is trace preserving, $\|x' - \Theta(x')\|_2 < 2\varepsilon$. On the other hand $\Theta^m(x') \in N'_{k-2m} \cap M_{j+2m}$ for all m and $(N'_{k-2m} \cap M_{j+2m}) \cap (N'_k \cap M_j) = \mathbf{C}1$ for a large enough m. Hence $x \in \mathbf{C}1$. Assume that $N \subset M$ has finite depth. Then Θ is a 2-shift on the tower $(M'_{-k} \cap M_j)_{k,j}$ for R by the same proof as one for Γ. Since $M'_{-k} \cap M_j$ is isomorphic to A_{j+k}, we have by Theorem 1,

$$H(\Theta) = \lim_{k\to\infty} \frac{H(M'_{-k} \cap M_k)}{k} = \lim_k \frac{H(M' \cap M_{2k})}{k} = H(\Gamma).$$

Hence we have the relation by Corollary 24. □

Acknowledgments

The author would like to express her thanks for the hospitality of the organizers U. Haagerup and E. Størmer at the Mittag-Leffler Institute and also A. Connes and M. Takesaki at the Institute des Hautes Etudes Scientifiques during the spring of 1989.

References

1. M. Choda, *Shifts on the hyperfinite* II_1*-factor*, J. Operator Theory **17** (1987), 223–235.
2. ——, *Entropy for *-endomorphisms and relative entropy for subalgebras*, J. Operator Theory (to appear).
3. A. Connes and E. Størmer, *Entropy for automorphism of* II_1 *von Neumann algebras*, Acta Math. **134** (1975), 288–306.
4. A Connes, H. Narnhofer, and W. Thirring, *Dynamical entropy of C^*-algebras and von Neumann algebras*, Comm. Math. Phys. **112** (1987), 691–719.
5. F. Goodman, P. de la Harpe, and V. Jones, *Coxeter graphs and towers of algebras*, Math. Sci. Res. Inst. Publ. 14, Springer-Verlag, 1989.
6. V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1-25.
7. R. Longo, *Simple injective subfactors*, Adv. in Math. **63** (1987), 152–171.
8. ——, *Solution of the factorial Stone-Weierstrass conjecture. An application of the theory of standard split W^*-inclusions*, Invent. Math. **76** (1984), 145–155.
9. A. Ocneanu, *Quantized groups, string algebras, and Galois theory for algebras*, Preprint.
10. M. Pimsner and S. Popa, *Entropy and index for subfactors*, Ann. Sci. École Norm. Sup. **19** (1986), 57–106.
11. ——, *Iterating the basic construction*, Trans. Amer. Math. Soc. **310** (1988), 127–134.
12. S. Popa, *Maximal injective subalgebras in factors associated with free groups*, Adv. in Math. **50** (1983), 27–48.
13. ——, *Classification of subfactors: The reduction to commuting squares*, Preprint.

14. R. T. Powers, *An index theory for semigroups of *-endomorphisms of $B(H)$ and type* II_1-*factors*, Canad. J. Math. **40** (1988), 86–114.

15. M. Takesaki, *Theory of operator algebras*. I, Springer-Verlag, 1979.

16. H. Umegaki, *Conditional expectations in an operator algebra*, Tôhoku Math. J. **6** (1954), 358–362.

17. H. Wenzl, *Representation of Hecke algebras and subfactors*, Thesis, Univ. of Pennsylvania, 1985.

DEPARTMENT OF MATHEMATICS, OSAKA KYOIKU UNIVERSITY, TENNOJI, OSAKA 543 JAPAN
E-mail address: f00527@sinet.ad.jp